Why can you find animals with spotted coats and striped tails but no animals with striped bodies and spotted tails?

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The answer involves mathematics.
The answer involves a LOT of mathematics!
What is mathematics?

- Pre-500 BC: Use of arithmetic, some geometry and trigonometry.
- 500 BC–300 AD: Detailed study of number and shape.
- 17th Century: the study of number, shape, and motion (calculus).
- 20th Century: the study of patterns.
What kinds of pattern?

- Counting (numbers, arithmetic)
- Numbers (number theory)
- Shape (geometry)
- Measuring (e.g. trigonometry)
- Motion and change (calculus)
- Putting things together (algebra)
- Chance events (probability theory)
- etc.
How do we see these kinds of pattern?

Mathematics is a language for describing abstract patterns. When we use it to do that, it gives rise to the "Science of (abstract) Patterns".

Sometimes with our eyes ... but mostly with our minds — through mathematics.
What do we use this mathematical language for?

To understand our world (and ourselves) and use that understanding to do things in the world.
Who is this?

and what is he famous for?
Galileo (1564 – 1642)

The inventor of modern science

“To understand the universe, you have to understand the language in which it is written. That language is mathematics.”
How does this language called mathematics help us to understand our world?

It makes the invisible visible.
Making the invisible visible
More technologies that make the invisible visible
Making the invisible visible using mathematics
Why are there all those symbols?

You need an abstract notation to describe abstract structures and patterns precisely.
Why are there all those symbols?

\[ A_i^n = \frac{1}{4} \varepsilon_{ijk} P Q_j^n P Q_k^{n+1} \varepsilon_{lmn} P Q_l^n P Q_m^{n+1}, \]

where \( \varepsilon_{ijk} \) is the permutation symbol. Using the Kronecker delta \( \delta_{ij} \), and using \( \frac{\partial P}{\partial P} = \delta_{ij} \) as well as \( \nabla = \partial/\partial P \), we derive:

\[
\frac{\partial A_i}{\partial P^q} = 2 A_i \frac{\partial A_i}{\partial P^q} - \varepsilon_{ijk} \varepsilon_{ilm} \left[ -\delta_{jk} P Q_j^{n+1} P Q_k^n P Q_l^{n+1} - \delta_{ik} P Q_i^{n+1} P Q_l^n P Q_m^{n+1} - \delta_{il} P Q_j^n P Q_k^{n+1} P Q_l^{n+1} - \delta_{im} P Q_j^n P Q_k^{n+1} P Q_l^{n+1} \right]
\]

Using the \( \varepsilon \delta \) rule stating \( \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jm} \delta_{ik} - \delta_{jm} \delta_{ik} \), we obtain:

\[
\frac{\partial A_i}{\partial P^q} = \frac{1}{2} \left[ -||PQ^{n+1}||^2 P Q_i^n + (PQ^n \cdot PQ^{n+1}) P Q_i^{n+1} \right]
\]

Consequently:

\[
\frac{\partial A_i}{\partial P} = \frac{1}{4 A_i} \left( (PQ^{n+1} \cdot Q^{n+1} Q^n) P Q^n + (PQ^n \cdot Q^n Q^{n+1}) P Q^{n+1} \right).
\]

Using Eqn. (13), we find:

\[
\frac{\nabla A}{2 A} = \frac{1}{2 A} \sum_i \frac{\partial A_i}{\partial P}
\]
What are these abstract symbols good for?

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla \rho \text{ Euler's equation} \]

\[ \mathbf{u} = -\nabla \phi \quad \text{so } \nabla \times \mathbf{u} = 0 \quad \text{irrotational} \]

\[ \mathbf{F} = -\nabla \Phi \quad \text{conservative} \]

\[ \rho = \text{const. or } f(\rho) \quad \text{incompressible} \]

\[ \frac{\partial}{\partial t} (-\nabla \phi) + \nabla \phi \cdot \nabla \nabla \phi = -\nabla \Phi - \frac{1}{\rho} \nabla \rho \]

\[ \nabla \left[ -\frac{\partial \phi}{\partial t} + \frac{\nabla^2 \phi}{2} + \phi + \frac{p}{\rho} \right] = 0 \]

\[ -\frac{\partial \phi}{\partial t} + \frac{\nabla^2 \phi}{2} + \phi + \frac{p}{\rho} = C \]

\[ \frac{\nabla^2 \phi}{2} + \phi + \frac{p}{\rho} = C \quad \text{Bernoulli's equation} \]

Bernoulli's Equation
They let us “see” one of the forces that keep this in the air.
What are these abstract symbols good for?

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{F} - \frac{1}{\rho} \nabla p \quad \text{Euler's equation} \]

\[ \mathbf{v} = -\nabla \phi \quad \text{so} \quad \nabla \times \mathbf{v} = 0 \quad \text{irrotational} \]

\[ \mathbf{F} = -\nabla \Omega \quad \text{conservative} \]

\[ \rho = \text{const. or } f(p) \quad \text{incompressible} \]

\[ \frac{\partial}{\partial t} (-\nabla \phi) + \nabla \phi \cdot \nabla \phi = -\nabla \Omega - \frac{1}{\rho} \nabla p \]

\[ \nabla \left[ -\frac{\partial \phi}{\partial t} + \frac{\nu^2}{2} + \Omega + \frac{p}{\rho} \right] = 0 \]

\[ -\frac{\partial \phi}{\partial t} + \frac{\nu^2}{2} + \Omega + \frac{p}{\rho} = C \]

Bernoulli's equation

\[ \frac{\nu^2}{2} + \Omega + \frac{p}{\rho} = C \]

Bernoulli's Equation
They let us “see” one of the forces that keep this in the air.
How does this store music?

Using this mathematics

Fourier series
What do these equations help us to understand?

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p = \mu \Delta \mathbf{u} + \mathbf{f},
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

\[
\rho(x, t) = \int M(q, r, s) \delta (x - X(q, r, s, t)) \, dq \, dr \, ds,
\]

\[
f(x, t) = \int F(q, r, s, t) \delta (x - X(q, r, s, t)) \, dq \, dr \, ds,
\]

\[
\frac{\partial X}{\partial t}(q, r, s, t) = \mathbf{u}(X(q, r, s, t), t)
\]

\[
= \int \mathbf{u}(x, t) \delta (x - X(q, r, s, t)) \, dx,
\]

\[
F = -\frac{\rho E}{\rho X}.
\]
These equations help us understand how the human heart works.

\[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) + \nabla p = \mu \Delta u + f, \]
\[ \nabla \cdot u = 0, \]
\[ \rho(x, t) = \int M(q, r, s) \delta(x - X(q, r, s, t)) \, dq \, dr \, ds, \]
\[ f(x, t) = \int F(q, r, s, t) \delta(x - X(q, r, s, t)) \, dq \, dr \, ds, \]
\[ \frac{\partial X}{\partial t}(q, r, s, t) = u(X(q, r, s, t), t) \]
\[ = \int u(x, t) \delta(x - X(q, r, s, t)) \, dx, \]
\[ F = -\frac{\rho E}{\rho X}. \]
What do we see with this?

\[ \dot{x} = f_1 = u \]
\[ \dot{u} = f_2 = 2v + x - \frac{\mu(-1 + x + \mu)}{(y^2 + z^2 + (-1 + x + \mu)^2)^{3/2}} - \frac{(1 - \mu)(x + \mu)}{(y^2 + z^2 + (x + \mu)^2)^{3/2}} \]
\[ \dot{y} = f_3 = v \]
\[ \dot{v} = f_4 = -2u + y - \frac{y\mu}{(y^2 + z^2 + (-1 + x + \mu)^2)^{3/2}} - \frac{y(1 - \mu)}{(y^2 + z^2 + (x + \mu)^2)^{3/2}} \]
\[ \dot{z} = f_5 = w \]
\[ \dot{w} = f_6 = -\frac{z\mu}{(y^2 + z^2 + (-1 + x + \mu)^2)^{3/2}} - \frac{z(1 - \mu)}{(y^2 + z^2 + (x + \mu)^2)^{3/2}} \]

This describes the forces that act on a body in outer space.
What do we see with this?

The mathematical equations of space flight:

This describes the forces that act on a body in outer space.
Who uses these equations?

\[ A_i^2 = \frac{1}{4} \varepsilon_{ijk} P Q_j^m P Q_k^{m+1} \varepsilon_{imn} P Q_i^m P Q_m^{n+1}, \]

where \( \varepsilon_{ijk} \) is the permutation symbol. Using the Kronecker delta \( \delta_{ij} \), and using \( \frac{\partial P}{\partial \eta} = \delta_{iq} \) as well as \( \nabla = \partial / \partial \eta \), we derive:

\[
\frac{\partial A_i^2}{\partial \eta} = 2 A_i \frac{\partial A_i}{\partial \eta} = \frac{1}{4} \varepsilon_{ijk} \varepsilon_{imn} \left[ -\delta_{jt} P Q_j^{n+1} P Q_j^m P Q_m^{n+1} - \delta_{jt} P Q_j^m P Q_j^{n+1} P Q_m^{n+1} - \delta_{jt} P Q_j^m P Q_j^{n+1} P Q_m^{n+1} \right]
\]

Using the \( \varepsilon \)-\( \delta \) rule stating \( \varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \), we obtain:

\[
\frac{\partial A_i^2}{\partial \eta} = \frac{1}{2} \left[ -||P Q^{n+1}||^2 P Q^n + (P Q^n \cdot P Q^{n+1}) P Q^{n+1} - ||P Q^n||^2 P Q^{n+1} + (P Q^{n+1} \cdot P Q^n) P Q^n \right] \eta
\]

\[
= \frac{1}{2} \left[ (P Q^{n+1} \cdot Q^{n+1} Q^n) P Q^n + (P Q^n \cdot Q^n Q^{n+1}) P Q^{n+1} \right] \eta.
\]

Consequently:

\[
\frac{\partial A_i}{\partial \eta} = \frac{1}{4 A_i} \left( (P Q^{n+1} \cdot Q^{n+1} Q^n) P Q^n + (P Q^n \cdot Q^n Q^{n+1}) P Q^{n+1} \right).
\]

Using Equ. (13), we find:

\[
\frac{\nabla A}{2 A} = \frac{1}{2 A} \sum_i \frac{\partial A_i}{\partial \eta}.
\]

(16)
Who uses these equations?

\[ A^2 = \frac{1}{4} \varepsilon_{ijk} P Q^i_k P Q^{i+1}_k \varepsilon_{ilm} P Q^l_m P Q^{l+1}_m, \]

where \( \varepsilon_{ijk} \) is the permutation symbol. Using the Kronecker delta \( \delta_{ij} \), and using \( \frac{\partial P}{\partial P_q} = \delta_{q} \) as well as \( \nabla = \partial / \partial P_q \), we derive:

\[
\frac{\partial A^2}{\partial P_q} = 2 A_i \frac{\partial A_i}{\partial P_q},
\]

\[
= \frac{1}{4} \varepsilon_{ijk} \varepsilon_{ilm} \left[ - \delta_{ij} P Q^{i+1}_k P Q^j_l P Q^{i+1}_m P Q^j_l - \delta_{ij} P Q^j_l P Q^{i+1}_m P Q^j_l - \delta_{ij} P Q^j_l P Q^{i+1}_m P Q^j_l - \delta_{ij} P Q^{i+1}_m P Q^j_l P Q^{i+1}_m P Q^j_l \right]
\]

Using the \( \varepsilon - \delta \) rule stating \( \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \), we obtain:

\[
\frac{\partial A^2}{\partial P_q} = \frac{1}{2} \left[ - ||P Q^{i+1}||^2 P Q^i + (P Q^i \cdot P Q^{i+1}) P Q^{i+1} \right.
\]

\[
- ||P Q^i||^2 P Q^{i+1} + (P Q^{i+1} \cdot P Q^i) P Q^i \left. \right]_q
\]

\[
= \frac{1}{2} \left[ (P Q^{i+1} \cdot Q^{i+1} Q^i) P Q^i + (P Q^i \cdot Q^i Q^{i+1}) P Q^{i+1} \right]_q.
\]

Consequently:

\[
\frac{\partial A_i}{\partial P} = \frac{1}{4 A_i} \left( (P Q^{i+1} \cdot Q^{i+1} Q^i) P Q^i + (P Q^i \cdot Q^i Q^{i+1}) P Q^{i+1} \right).
\]

Using Eqn. (13), we find:

\[
\frac{\nabla A}{2 A} = \frac{1}{2 A} \sum_i \frac{\partial A_i}{\partial P}
\]

(16)
Who uses these equations?

\[ A_i^2 = \frac{1}{4} \varepsilon_{ijk} P Q_j^m P Q_k^{m+1} \varepsilon_{ilm} P Q_i^m P Q_l^{m+1}, \]

where \( \varepsilon_{ijk} \) is the permutation symbol. Using the Kronecker delta \( \delta_{ij} \), and using \( \frac{\partial P}{\partial Q} = \delta_{ij} \) as well as \( \nabla = \partial/\partial P \), we derive:

\[
\frac{\partial A_i^2}{\partial P_q} = 2 A_i \frac{\partial A_i}{\partial P_q} = \frac{1}{4} \varepsilon_{ijk} \varepsilon_{ilm} \left[ -\delta_{ij} P Q_j^m P Q_k^{m+1} P Q_l^{m+1} - \delta_{ij} P Q_i^m P Q_j^{m+1} P Q_l^{m+1} \\
- \delta_{ij} P Q_i^m P Q_k^{m+1} P Q_l^{m+1} - \delta_{ij} P Q_i^m P Q_k^{m+1} P Q_l^{m+1} \right]
\]

Using the \( \varepsilon - \delta \) rule stating \( \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \), we obtain:

\[
\frac{\partial A_i^2}{\partial P_q} = \frac{1}{2} \left[ -||P Q_i^{m+1}||^2 P Q_i^m + (P Q_i^m P Q_i^{m+1}) P Q_i^{m+1} \\
-||P Q_i^m||^2 P Q_i^{m+1} + (P Q_i^{m+1} P Q_i^m) P Q_i^{m+1} \right] q
\]

Consequently:

\[
\frac{\partial A_i}{\partial P} = \frac{1}{4 A} \left( (P Q_i^{m+1} P Q_i^m + (P Q_i^m P Q_i^{m+1}) P Q_i^{m+1}) \right).
\]

Consequently:

\[
\frac{\partial A_i}{\partial P} = \frac{1}{4 A} \left( (P Q_i^{m+1} P Q_i^m + (P Q_i^m P Q_i^{m+1}) P Q_i^{m+1}) \right).
\]

Using Equ. (13), we find:

\[
\frac{\nabla A}{2 A} = \frac{1}{2 A} \sum_i \frac{\partial A_i}{\partial P}
\]

(16)
What is this math used for?

\[ p_{ij} = k \sum_{n=1}^{c} \left[ \frac{\phi}{(|x_i - x_n| + |y_j - y_n|)^f} + \frac{(1 - \phi)(B^{g-f})}{(2B - |x_i - x_n| - |y_j - y_n|)^g} \right] \]
Catching criminals

\[ P_0 = k \sum_{i} \frac{\phi}{(x_i - x_n) + (y_j - y_n)} + \frac{(1 - \phi)(B^{\epsilon})}{(2B - |x_i - x_n| - |y_j - y_n|)^\epsilon} \]
NUMB3RS: First ever episode

\[ p_{ij} = k \sum_{n=1}^{c} \left[ \frac{\phi}{(|x_i - x_n| + |y_j - y_n|)^{r}} + \frac{(1 - \phi)(B^{e-f})}{(2B - |x_i - x_n| - |y_j - y_n|)^{k}} \right] \]
My examples show patterns of:

- Forces keeping and aircraft in the air
- Musical sounds (iPod)
- The human heartbeat
- Forces acting on a spacecraft
- The structure of things we see (movie graphics)
- Criminal behavior

Time to look at ...
Animal coat patterns

[Images of various animals with different coat patterns, including a tiger, a leopard, a zebra, and a snow leopard.]
Some animals don’t have patterns
Some do have patterns

What are the most interesting patterns?

- We see the skin “pattern” with our eyes.
- Can we see use mathematics to see (with our minds) what creates those patterns of spots and stripes?
- Can we see nature’s invisible pattern?
How do the coat patterns arise?

- Skin color is caused by a chemical called melanin.
- The skin pattern is a result of different concentrations of melanin.
- What makes the melanin concentrate the way it does?
- The basic mechanism is believed to be a "reaction-diffusion" process.
Reaction-diffusion process

Computer simulation
Reaction-diffusion process

Computer simulation
Reaction-diffusion process

Computer simulation
Reaction-diffusion process

Computer simulation
How we think it works

- Melanin production in the skin is initiated or sped-up by an “activator” chemical, and slowed down or stopped by an “inhibitor” chemical.
- The activator and inhibitor create a reaction-diffusion process.
- The inhibitor spreads faster.
- The initial distribution of the activator and inhibitor is random.
What makes the patterns different?

- The area and shape of the skin during the reaction.
- Small areas allow no space for the diffusion, so there is no pattern.
- With a large area the inhibitor eventually occupies the entire area, so again no pattern.
- Thus mice and elephants have neither stripes or spots.
- In a long, thin rectangular area, the inhibitor and activator will form alternating bands, so you get stripes.
- In a squarish area, the inhibitor will surround areas of activator so you get spots.

James Murray
When it happens

For most creatures, the key reaction-diffusion process takes place during the embryonic stage.

So their final coat pattern depends on the area and shape of the embryo, not the adult creature.
Animal coat patterns
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Animal coat patterns
Bodies and tails

Murray’s computer simulations
MORE BY ME ABOUT THESE IDEAS

Life by the Numbers
Six part television series
PBS: WQED-tv 1998
available from
http://www.montereymedia.com/science

MATHMATICS
The Science of Patterns
KEITH DEVLIN

THE LANGUAGE OF
Mathematics
Making the invisible visible
KEITH DEVLIN

SOLVING CRIME WITH MATHEMATICS
THE NUMBERS BEHIND NUMB3RS
KEITH DEVLIN AND GARY LORDEN